A new collocation scheme for solving hyperbolic equations of second order in a semi-infinite domain

R.M. Hafez[1], M.A. Abdelkawy[2], E.H. Doha[3], A.H. Bhrawy[4,5,*]

[1]Department of Basic Science, Institute of Information Technology, Modern Academy, Cairo, Egypt
[2]Department of Mathematics and Computer Science, Faculty of Science, Beni-Suef University, Beni-Suef 62511, Egypt
[3]Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt
[4]Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia, *email: alibhrawy@yahoo.co.uk
[5]Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef 62511, Egypt
*Corresponding author: alibhrawy@yahoo.co.uk

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This paper reports a new fully collocation algorithm for the numerical solution of hyperbolic partial differential equations (PDEs) of second order in a semi-infinite domain, using Jacobi rational Gauss-Radau collocation (JR-GR-C) method. The widely applicable, efficiency, and high accuracy are the more advantages of the collocation method. The series expansion in Jacobi rational functions is the main step for solving the mentioned problems. The expansion coefficients are then determined by reducing the hyperbolic equations with its boundary and initial conditions to a system of algebraic equations for these coefficients. This system may be solved analytically or numerically in step-by-step manner by using Newtons iterative method. Numerical results are consistent with the theoretical analysis and indicating the high accuracy and effectiveness of this algorithm.

Key words: Hyperbolic PDEs; Jacobi rational functions; Collocation method; Semi-infinite domain; Gauss-Radau quadrature..

1. Introduction

Several problems in science and engineering fields are discussed in semi-infinite domains. The earthquake engineering field and underwater acoustic problems can be modeled as semi-infinite domain PDEs. Spectral methods based on specific polynomials (Laguerre, Hermite, rational Legendre polynomials $\cdots$ etc) [1]-[7] can be used to numerically solve problems on semi-infinite domains. The mapping problem in an unbounded domain to that in a bounded domains has been used in [8]-[11] to approximate the problems in unbounded domain. For more details about numerical solutions for unbounded domain problems, see for example [12]- [21].

In recent years there has been a high level of interest of employing spectral methods for numerically solving many types of integral and differential equations, due to their ease of applying them for finite and infinite domains [22]-[30]. The speed of convergence is one of the great advantages of spectral method. Besides, that the spectral methods have exponential rates of convergence; they also have high level of accuracy [31]-[37] method. The main idea of all versions of spectral methods is to express the spectral solution of the problem as a finite sum of certain basis functions (orthogonal polynomials or combination of orthogonal polynomials) and then to choose the coefficients in order to minimize the difference between the exact and numerical solutions as well as possible. The spectral collocation method is a specific type of spectral methods, that is more applicable and widely used to solve almost types of differential equations.

In the present paper, we numerically solve two hyperbolic equations of second order. The main aim of this paper is to extend the application of JR-GR-C scheme for the numerical solutions of the hyperbolic equations in semi-infinite domain. The solution $u(x,t)$ of such equation is approximated as $u_N(x,t)$ which can be expressed as a finite expansion of Jacobi rational polynomials for the space and time variables, and then we evaluate the partial derivatives of finite expansion of $u_N(x,t)$ at the Jacobi rational Gauss-Radau quadrature points. Substituting these approximations in the underlined equation provides a system of algebraic equations. This system may be solved analytically or numerically by Newtons iterative method. This scheme is one of the suitable methods for solving system of algebraic equations.

The outline of this paper is arranged as follows. We present few revelent properties of Jacobi rational polynomials in the coming section. In Section 3, we introduce JR-GR-C method for hyperbolic equations of second order in semi-infinite domain. Numerical examples and simulations are presented in Section 4 to show the effectiveness and accuracy of the underlying method. In the last section, we present some observations and conclusions.

2. Preliminaries

The standard Jacobi polynomial of degree k ($P_k^{(\alpha,\beta)}(x)$, $k = 0,1,\cdots$) with the parameters $\alpha > -1$, $\beta > -1$, satisfy the following relations

$$\begin{aligned} P_k^{(\alpha,\beta)}(-x) &= (-1)^k P_k^{(\alpha,\beta)}(x), \\ P_k^{(\alpha,\beta)}(-1) &= \frac{(-1)^k\Gamma(k+\beta+1)}{k!\Gamma(\beta+1)}, \\ P_k^{(\alpha,\beta)}(1) &= \frac{\Gamma(k+\alpha+1)}{k!\Gamma(\alpha+1)}. \end{aligned} \tag{1}$$

Let $w^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}$, then we define the weighted space $L^2_{w^{(\alpha,\beta)}}$ as usual, equipped with the following inner product and norm,

$$(u,v)_{w^{(\alpha,\beta)}} = \int_{-1}^{1} u(x)\,v(x) w^{(\alpha,\beta)}(x)dx, \qquad \|u\|_{w^{(\alpha,\beta)}} = (u,u)^{\frac{1}{2}}_{w^{(\alpha,\beta)}}. \tag{2}$$

The set of Jacobi polynomials forms a complete $L^2_{w^{(\alpha,\beta)}}$-orthogonal system, and

$$\|P_k^{(\alpha,\beta)}\|_{w^{(\alpha,\beta)}} = h_k = \frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}. \tag{3}$$

Let $R_k^{(\alpha,\beta)}(x),\ x\in[0,\infty[$ be the Jacobi rational functions defined by (cf. [39])

$$R_k^{(\alpha,\beta)}(x) = P_k^{(\alpha,\beta)}(\frac{x-1}{x+1}), \quad k = 0,1,2,\ldots,$$

where $P_k^{(\alpha,\beta)}(.)$ is the Jacobi polynomial of degree k defined on $[-1,1]$. From the standard properties of Jacobi polynomials, one can easily deduce that

$$(k+\alpha+1)R_k^{(\alpha,\beta)}(x) - (k+1)R_{k+1}^{(\alpha,\beta)}(x) = (2k+\alpha+\beta+2)(x+1)^{-1}R_k^{(\alpha+1,\beta)}(x),$$

$$R_k^{(\alpha,\beta)}(x) = (-1)^k R_k^{(\beta,\alpha)}(\frac{1}{x}), \quad R_k^{(\alpha,\beta)}(\infty) = \frac{\Gamma(k+\alpha+1)}{k!\Gamma(\alpha+1)},$$

$$\begin{aligned} D^q R_k^{(\alpha,\beta)}(0) = &\sum_{f=0}^{q-1}(-1)^f \begin{pmatrix} q \\ f \end{pmatrix} \frac{(q-1)!\Gamma(k+\alpha+\beta+q-f+1)}{(q-f-1)!(k-q+f)!} \\ &\times \frac{\Gamma(k+\beta+1)}{\Gamma(k+\alpha+\beta+1)\Gamma(\beta+q-f+1)}, \end{aligned} \tag{4}$$

$$(k+\alpha+1)R_k^{(\alpha,\beta)}(x) - (k+1)R_{k+1}^{(\alpha,\beta)}(x) = (2k+\alpha+\beta+2)(x+1)^{-1}R_k^{(\alpha+1,\beta)}(x),$$

$$R_k^{(\alpha,\beta-1)}(x) - R_k^{(\alpha-1,\beta)}(x) = R_{k-1}^{(\alpha,\beta)}(x),$$

$$(k+\alpha+\beta)R_k^{(\alpha,\beta)}(x) = (k+\beta)R_k^{(\alpha,\beta-1)}(x) + (k+\alpha)R_k^{(\alpha-1,\beta)}(x),$$

and

$$\begin{aligned} D^q R_k^{(\alpha,\beta)}(x) = &\sum_{f=0}^{q-1}(-1)^f \begin{pmatrix} q \\ f \end{pmatrix} \frac{(q-1)!}{(q-f-1)!}(x+1)^{-(2q-f)} \\ &\frac{\Gamma(k+\alpha+\beta+q-f+1)}{\Gamma(k+\alpha+\beta+1)} R_{k-q+f}^{(\alpha+q-f,\beta+q-f)}(x). \end{aligned} \tag{5}$$

Next, let $\chi_R^{(\alpha,\beta)}(x) = x^{\beta}(x+1)^{-\alpha-\beta-2}$. Then for $\alpha,\beta > -1$, the set of Jacobi rational functions is a complete $L^2_{\chi_R^{(\alpha,\beta)}}[0,\infty)$-orthogonal system, i.e.,

$$\int_0^{\infty} R_k^{(\alpha,\beta)}(x) R_l^{(\alpha,\beta)}(x) \chi_R^{(\alpha,\beta)}(x) dx = \gamma_k^{(\alpha,\beta)} \delta_{k,l},$$

where

$$\gamma_k^{(\alpha,\beta)} = \frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}. \tag{6}$$

We now turn to the Jacobi-Gauss interpolation. We denote by $x_{N,j}^{(\alpha,\beta)}$, $0 \leqslant j \leqslant N$, the nodes of the standard Jacobi-Gauss interpolation on the interval $(-1,1)$. Their corresponding Christoffel numbers are $\varpi_{N,j}^{(\alpha,\beta)}$, $0 \leqslant j \leqslant N$. The nodes of the Jacobi rational-Gauss interpolation on the interval $(0,\infty)$ are the zeros of $R_{N+1}^{(\alpha,\beta)}(x)$, which we denote by $x_{R,N,j}^{(\alpha,\beta)}$, $0 \leqslant j \leqslant N$. Clearly $x_{R,N,j}^{(\alpha,\beta)} = \frac{1+x_{N,j}^{(\alpha,\beta)}}{1-x_{N,j}^{(\alpha,\beta)}}$, and their corresponding Christoffel numbers are $\varpi_{R,N,j}^{(\alpha,\beta)} = \frac{1}{2^{\alpha+\beta+1}}\, \varpi_{N,j}^{(\alpha,\beta)}$, $0 \leqslant j \leqslant N$.

Let N be any positive integer, and

$$S_N(0,\infty) = \text{span}\{R_0^{(\alpha,\beta)}(x), R_1^{(\alpha,\beta)}(x), \ldots, R_N^{(\alpha,\beta)}(x)\}. \tag{7}$$

It follows that for any $\phi \in S_{2N+1}(0,\infty)$,

$$\begin{aligned}
\int\limits_0^{\infty} x^{\beta}(x+1)^{-\alpha-\beta-2}\phi(x)dx &= \frac{1}{2^{\alpha+\beta+1}} \int\limits_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}\phi\left(\frac{1+x}{1-x}\right)dx \\
&= \frac{1}{2^{\alpha+\beta+1}} \sum_{j=0}^{N} \varpi_{N,j}^{(\alpha,\beta)} \phi\left(\frac{1+x_{N,j}^{(\alpha,\beta)}}{1-x_{N,j}^{(\alpha,\beta)}}\right) \\
&= \sum_{j=0}^{N} \varpi_{R,N,j}^{(\alpha,\beta)} \phi\left(x_{R,N,j}^{(\alpha,\beta)}\right).
\end{aligned} \tag{8}$$

In order to present the approximation results precisely, we introduce the space $H^r_{\chi_R^{(\alpha,\beta)},A}(\Lambda)$, $r \in \mathbb{N}$, $\Lambda \equiv (0,\infty)$ with the following semi-norm and norm:

$$|\upsilon|_{r,\chi_R^{(\alpha,\beta)},A} = (\sum_{k=r}^{\infty} (\lambda_k^{(\alpha,\beta)})^r |a_k|^2 \gamma_k^{(\alpha,\beta)})^{\frac{1}{2}}, \quad \|\upsilon\|_{r,\chi_R^{(\alpha,\beta)},A} = (\sum_{l=0}^{r} |\upsilon|^2_{l,\chi_R^{(\alpha,\beta)},A})^{\frac{1}{2}}. \tag{9}$$

For any $r > 0$, we define the space $H^r_{\chi_R^{(\alpha,\beta)},A}(\Lambda)$ and its norm by space interpolation as in [38, 39].

Theorem 2.1 *For any* $\upsilon \in H^r_{\chi_R^{(\alpha,\beta)},A}(\Lambda)$, $r \in \mathbb{N}$ *and* $0 \le \mu \le r$,

$$\|\mathbf{P}_{N,\alpha,\beta}\upsilon - \upsilon\|_{\mu,\chi_R^{(\alpha,\beta)},A} \le CN^{\mu-r}|\upsilon|_{r,\chi_R^{(\alpha,\beta)},A}. \tag{10}$$

A complete proof of the theorem and discussion on convergence are given in [39].

3. Fully JR-GR-C method

This section presents a new collocation method for numerically solving the one-dimensional hyperbolic equation of second order in semi-infinite domain:

$$\partial_{tt}u(x,t) = H(x,t,u(x,t),\partial_x u(x,t),\partial_{xx}u(x,t)), \quad (x,t) \in [0,\infty)\times[0,\infty), \tag{11}$$

subject to the conditions (initial conditions and conditions at infinity)

$$\begin{aligned} &u(x,0) = g_0(x), \quad \lim_{t\to\infty}\partial_t u(x,t) = g_1(x), \quad x \in [0,\infty),\\ &u(0,t) = g_2(t), \quad \lim_{x\to\infty}\partial_x u(x,t) = g_3(t), \quad t \in [0,\infty), \end{aligned} \tag{12}$$

where $H(x,t,u(x,t),\partial_x u(x,t),\partial_{xx}u(x,t))$, $g_0(x)$, $g_1(x)$, $g_2(t)$ and $g_3(t)$ are given functions.

In the proposed collocation method, two sets of Jacobi rational Gauss-Radau points, with two different Jacobi rational parameters, are adopted for the spatial and temporal discretizations. Now, we outline the main steps of the JR-GR-C method for solving one-dimensional hyperbolic equation of second order. Let us assume the approximate solution has the form

$$\begin{aligned} u(x,t) &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} R_i^{(\alpha_1,\beta_1)}(x) R_j^{(\alpha_2,\beta_2)}(t)\\ &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(x,t), \end{aligned} \tag{13}$$

where

$$f_0^{i,j}(x,t) = R_i^{(\alpha_1,\beta_1)}(x) R_j^{(\alpha_2,\beta_2)}(t).$$

We can approximate the spatial partial derivative $\partial_x u(x,t)$ as

$$\begin{aligned} \partial_x u(x,t) &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j}\partial_x\left(R_i^{(\alpha_1,\beta_1)}(x)\right) R_j^{(\alpha_2,\beta_2)}(t)\\ &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_1^{i,j}(x,t), \end{aligned} \tag{14}$$

where

$$f_1^{i,j}(x,t) = \partial_x \left(R_i^{(\alpha_1,\beta_1)}(x) \right) R_j^{(\alpha_2,\beta_2)}(t).$$

Similarly, the approximation of the time partial derivative $\partial_t u(x,t)$ is

$$\begin{aligned} \partial_t u(x,t) &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} R_i^{(\alpha_1,\beta_1)}(x) \partial_t \left(R_j^{(\alpha_2,\beta_2)}(t) \right) \\ &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_2^{i,j}(x,t), \end{aligned} \tag{15}$$

where

$$f_2^{i,j}(x,t) = R_i^{(\alpha_1,\beta_1)}(x) \partial_t R_j^{(\alpha_2,\beta_2)}(t).$$

Furthermore, the approximations of the second spatial and temporal partial derivatives ($\partial_{xx} u(x,t)$ and $\partial_{tt} u(x,t)$) are

$$\begin{aligned} \partial_{xx} u(x,t) &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_3^{i,j}(x,t), \\ \partial_{tt} u(x,t) &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_4^{i,j}(x,t), \end{aligned} \tag{16}$$

where

$$f_3^{i,j}(x,t) = \partial_{xx}(R_i^{(\alpha_1,\beta_1)}(x)) R_j^{(\alpha_2,\beta_2)}(t),$$

and

$$f_4^{i,j}(x,t) = R_i^{(\alpha_1,\beta_1)}(x) \partial_{tt}(R_j^{(\alpha_2,\beta_2)}(t)).$$

Accordingally, adopting (13)-(16), enable one to write (11)-(12) in the form:

$$\begin{aligned} \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_4^{i,j}(x,t) = H\Big(x,t,& \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(x,t), \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_1^{i,j}(x,t), \\ & \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_3^{i,j}(x,t)\Big), \\ & (x,t) \in [0,\infty) \times [0,\infty), \end{aligned} \tag{17}$$

where the functions $f_1^{i,j}(x,t)$, $f_2^{i,j}(x,t)$, $f_3^{i,j}(x,t)$ and $f_4^{i,j}(x,t)$, are explicitly expressed by means of (5) at $q = 1, 2$, with some calculations at the Jacobi rational Gauss-Radau quadrature nodes.

The approximations of the boundary conditions (12) may be obtained from

$$
\begin{aligned}
&u(x,0) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(x,0) = g_0(x),\\
&\lim_{t\to\infty} u(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} R_i^{(\alpha_1,\beta_1)}(x)\frac{\Gamma(j+\alpha_2+1)}{j!\Gamma(\alpha_2+1)} = g_1(x),\\
&u(0,t) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(0,t) = g_2(t),\\
&\lim_{x\to\infty} u(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} R_j^{(\alpha_2,\beta_2)}(t)\frac{\Gamma(i+\alpha_1+1)}{i!\Gamma(\alpha_1+1)} = g_3(t).
\end{aligned}
\tag{18}
$$

In order to obtain the $(M+1)\times(N+1)$ unknowns namely, $a_{i,j}$ for the approximate solution (13). The residual of (17) is set equal to zero at $(N-1)\times(M-1)$ of JR-GR-C points. In addition, the approximations of boundary conditions in (18) are collocated at JR-GR-C points. Accordingly, we obtain $(N-1)\times(M-1)$ algebraic equations form

$$
\begin{aligned}
\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_4^{i,j}\left(x_{R,N,r}^{(\alpha_1,\beta_1)}, t_{R,M,s}^{(\alpha_2,\beta_2)}\right) =& H\left(x_{R,N,r}^{(\alpha_1,\beta_1)}, t_{R,M,s}^{(\alpha_2,\beta_2)}, \zeta_1^{r,s}, \zeta_2^{r,s}, \zeta_3^{r,s}\right),\\
& r=1,\cdots,N-1; \quad s=1,\cdots,M-1,
\end{aligned}
\tag{19}
$$

where

$$
\begin{aligned}
\zeta_1^{r,s} &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}\left(x_{R,N,r}^{(\alpha_1,\beta_1)}, t_{R,M,s}^{(\alpha_2,\beta_2)}\right),\\
\zeta_2^{r,s} &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_1^{i,j}\left(x_{R,N,r}^{(\alpha_1,\beta_1)}, t_{R,M,s}^{(\alpha_2,\beta_2)}\right),\\
\zeta_3^{r,s} &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_3^{i,j}\left(x_{R,N,r}^{(\alpha_1,\beta_1)}, t_{R,M,s}^{(\alpha_2,\beta_2)}\right).
\end{aligned}
$$

Due to the conditions at $t=0$ and $x=0$ in (12), we get an additional $(N-1)+(M+1)$ algebraic equations

$$
\begin{aligned}
\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(x_{R,N,r}^{(\alpha_1,\beta_1)},0) =& g_0(x_{R,N,r}^{(\alpha_1,\beta_1)}), \quad r=1,\cdots,N-1,\\
\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(0,t_{R,M,s}^{(\alpha_2,\beta_2)}) =& g_2(t_{R,M,s}^{(\alpha_2,\beta_2)}), \quad s=0,\cdots,M,
\end{aligned}
\tag{20}
$$

The spatial and temporal conditions at infinity provide $(N-1)+(M+1)$ algebraic equations

$$\begin{aligned}
&\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} R_i^{(\alpha_1,\beta_1)}(x_{R,N,r}^{(\alpha_1,\beta_1)})\frac{\Gamma(j+\alpha_2+1)}{j!\Gamma(\alpha_2+1)} = g_1(x_{R,N,r}^{(\alpha_1,\beta_1)}), \quad r=1,\cdots,N-1,\\
&\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} R_j^{(\alpha_2,\beta_2)}(t_{R,M,s}^{(\alpha_2,\beta_2)})\frac{\Gamma(i+\alpha_1+1)}{i!\Gamma(\alpha_1+1)} = g_3(t_{R,M,s}^{(\alpha_2,\beta_2)}), \quad s=0,\cdots,M.
\end{aligned} \tag{21}$$

This in turn, yields a system of $(M+1)\times(N+1)$ algebraic equations which may be written as

$$\begin{aligned}
&\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_4^{i,j}(x_{R,N,r}^{(\alpha_1,\beta_1)},t_{R,M,s}^{(\alpha_2,\beta_2)}) = H(x_{R,N,r}^{(\alpha_1,\beta_1)},t_{R,M,s}^{(\alpha_2,\beta_2)},\zeta_1^{r,s},\zeta_2^{r,s},\zeta_3^{r,s}),\\
&\qquad\qquad r=1,\cdots,N-1,\ s=1,\cdots,M-1,\\
&\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(x_{R,N,r}^{(\alpha_1,\beta_1)},0) = g_0(x_{R,N,r}^{(\alpha_1,\beta_1)}), \quad r=1,\cdots,N-1,\\
&\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(0,t_{R,M,s}^{(\alpha_2,\beta_2)}) = g_2(t_{R,M,s}^{(\alpha_2,\beta_2)}), \quad s=0,\cdots,M,\\
&\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} R_i^{(\alpha_1,\beta_1)}(x_{R,N,r}^{(\alpha_1,\beta_1)})\frac{\Gamma(j+\alpha_2+1)}{j!\Gamma(\alpha_2+1)} = g_1(x_{R,N,r}^{(\alpha_1,\beta_1)}), \quad r=1,\cdots,N-1,\\
&\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} R_j^{(\alpha_2,\beta_2)}(t_{R,M,s}^{(\alpha_2,\beta_2)})\frac{\Gamma(i+\alpha_1+1)}{i!\Gamma(\alpha_1+1)} = g_3(t_{R,M,s}^{(\alpha_2,\beta_2)}), \quad s=0,\cdots,M.
\end{aligned} \tag{22}$$

System (22) may also be written in the following matrix form

$$\begin{pmatrix}
\kappa_{1,1} & \cdots & \kappa_{1,M+1}\\
\kappa_{2,1} & \cdots & \kappa_{2,M+1}\\
\cdots & \ddots & \cdots\\
\cdots & \ddots & \cdots\\
\cdots & \ddots & \cdots\\
\kappa_{N,1} & \cdots & \kappa_{N,M+1}\\
\kappa_{N+1,1} & \cdots & \kappa_{N+1,M+1}
\end{pmatrix} = \begin{pmatrix}
\xi_{1,1} & \cdots & \xi_{1,M+1}\\
\xi_{2,1} & \cdots & \xi_{2,M+1}\\
\cdots & \ddots & \cdots\\
\cdots & \ddots & \cdots\\
\cdots & \ddots & \cdots\\
\xi_{N,1} & \cdots & \xi_{N,M+1}\\
\xi_{N+1,1} & \cdots & \xi_{N+1,M+1}
\end{pmatrix}, \tag{23}$$

where

$$\kappa_{l,m}=\begin{cases}\sum\limits_{i=0}^{N}\sum\limits_{j=0}^{M}a_{i,j}f_0^{i,j}(0,t_{R,M,m-1}^{(\alpha_2,\beta_2)}), & l=1,\quad m=1,\cdots,M+1,\\ \sum\limits_{i=0}^{N}\sum\limits_{j=0}^{M}a_{i,j}f_0^{i,j}(x_{R,N,l-1}^{(\alpha_1,\beta_1)},0), & m=1,\quad l=2,\cdots,N,\\ \sum\limits_{i=0}^{N}\sum\limits_{j=0}^{M}a_{i,j}R_j^{(\alpha_2,\beta_2)}(t_{R,M,m-1}^{(\alpha_2,\beta_2)})\frac{\Gamma(i+\alpha_1+1)}{i!\Gamma(\alpha_1+1)}, & l=N+1\quad m=1,\cdots,M+1,\\ \sum\limits_{i=0}^{N}\sum\limits_{j=0}^{M}a_{i,j}R_i^{(\alpha_1,\beta_1)}(x_{R,N,l-1}^{(\alpha_1,\beta_1)})\frac{\Gamma(j+\alpha_2+1)}{j!\Gamma(\alpha_2+1)}, & m=M+1\quad l=2,\cdots,N,\\ \sum\limits_{i=0}^{N}\sum\limits_{j=0}^{M}a_{i,j}f_4^{i,j}(x_{R,N,l-1}^{(\alpha_1,\beta_1)},t_{R,M,m-1}^{(\alpha_2,\beta_2)}), & l=2,\cdots,N-1,\ m=2,\cdots,M,\end{cases}\tag{24}$$

and

$$\xi_{l,m}=\begin{cases}g_2(t_{R,M,m-1}^{(\alpha_2,\beta_2)}), & l=1,\quad m=1,\cdots,M+1,\\ g_0(x_{R,N,l-1}^{(\alpha_1,\beta_1)}), & m=1,\quad l=2,\cdots,N,\\ g_3(t_{R,M,m-1}^{(\alpha_2,\beta_2)}), & l=N+1\quad m=1,\cdots,M+1,\\ g_1(x_{R,N,l-1}^{(\alpha_1,\beta_1)}), & m=M+1\quad l=2,\cdots,N,\\ \Omega, & l=2,\cdots,N-1,\ m=2,\cdots,M,\end{cases}\tag{25}$$

with

$$\Omega=H(x_{R,N,l-1}^{(\alpha_1,\beta_1)},t_{R,M,m-1}^{(\alpha_2,\beta_2)},\zeta_1^{l-1,m-1},\zeta_2^{l-1,m-1},\zeta_3^{l-1,m-1}).$$

4. Numerical examples

To illustrate the effectiveness of the proposed method, two test examples are considered. Comparison of the results obtained by various choices of Jacobi rational parameters α and β reveals that the new method is very accurate and efficient.

4.1. Exponential solution

Consider the hyperbolic PDE

$$\partial_{tt}u=\partial_{xx}u+u,\quad (x,t)\in[0,\infty)\times[0,\infty),\tag{26}$$

subject to the conditions

$$\begin{aligned}&u(0,t)=e^{-2t},\quad u(x,0)e^{-\sqrt{3}x},\\ &\lim_{x\to\infty}\partial_x u(x,t)=\lim_{t\to\infty}\partial_x u(x,t)=0.\end{aligned}\tag{27}$$

The exact solution of Eq. (26) is given by

$$u(x,t)=e^{-(2t+\sqrt{3}x)},\quad (x,t)\in[0,\infty)\times[0,\infty).\tag{28}$$

Table 1.

Maximum absolute errors using JR-GR-C method for equation (26)

$\alpha_1=\beta_1$	$\alpha_2=\beta_2$	4	8	12	16	20
0	0	1.68×10^{-2}	9.21×10^{-4}	7.56×10^{-5}	1.35×10^{-5}	1.87×10^{-6}
$-\frac{1}{2}$	$-\frac{1}{2}$	1.97×10^{-2}	1.95×10^{-3}	1.10×10^{-3}	2.61×10^{-4}	3.91×10^{-5}

Table 2.

Absolute errors using JR-GR-C method for equation (26)

x	t	E	x	t	E	x	t	E
0.1	0.1	6.98×10^{-7}	0.1	0.5	1.84×10^{-7}	0.1	1	2.36×10^{-7}
0.2		4.19×10^{-7}	0.2		6.22×10^{-9}	0.2		1.89×10^{-7}
0.3		2.58×10^{-7}	0.3		1.00×10^{-7}	0.3		1.56×10^{-7}
0.4		6.88×10^{-8}	0.4		2.89×10^{-8}	0.4		1.82×10^{-7}
0.5		1.72×10^{-7}	0.5		7.67×10^{-10}	0.5		1.49×10^{-7}
0.6		2.20×10^{-8}	0.6		1.10×10^{-8}	0.6		7.89×10^{-8}
0.7		1.20×10^{-7}	0.7		7.12×10^{-9}	0.7		2.03×10^{-8}
0.8		1.11×10^{-7}	0.8		4.56×10^{-8}	0.8		3.85×10^{-8}
0.9		1.66×10^{-8}	0.9		7.04×10^{-8}	0.9		9.27×10^{-8}

Maximum absolute errors of (26) subject to (27) are presented in Table 1, using the JR-GR-C method with two special values of Jacobi rational parameters $\alpha_1, \beta_1, \alpha_2, \beta_2$. It is clear that the special case $\alpha_1=\beta_1=\alpha_2=\beta_2=0$ (Legendre rational Gauss-Radau collocation method) is more accurate than $\alpha_1=\beta_1=\alpha_2=\beta_2=-\frac{1}{2}$ (the first kind Chebyshev rational Gauss-Radau collocation method). Meanwhile, absolute errors of problem (26) are presented in Table 2, for $\alpha_1=\beta_1=\alpha_2=\beta_2=0$ at $N=M=20$ with different values of (x,t).

Fig. 1 displays the absolute error of problem (26) with $\alpha_1=\beta_1=\alpha_2=\beta_2=0$ at $N=M=20$. From Fig. 2, we see that the curves of the approximate and exact solutions are coincided for different values of t. Meanwhile, the absolute error curve of the approximate solution of problem (26) at $t=50$ using JR-GR-C method with $\alpha_1=\beta_1=\alpha_2=\beta_2=0$ and $N=M=20$ is plotted in Fig. 3.

4.2. Solition solution

Finally, we consider the hyperbolic PDE

$$\partial_{tt}u=\partial_{xx}u+u+(\cosh(2(2t+x))-5)\operatorname{sech}^3(2t+x), \quad (x,t)\in[0,\infty)\times[0,\infty), \tag{29}$$

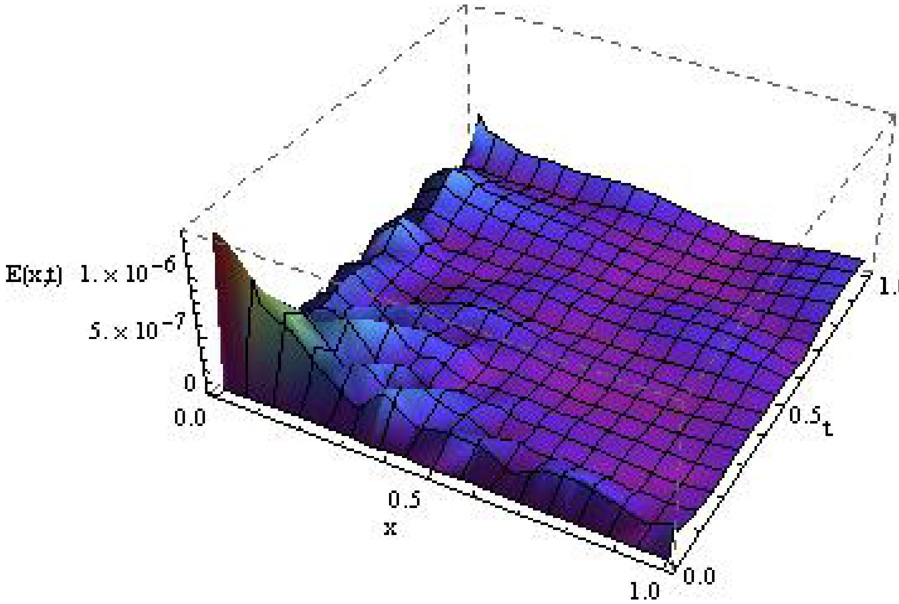

Fig. 1 – The absolute error of problem (26), using JR-GR-C method with $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$ at $N = M = 20$.

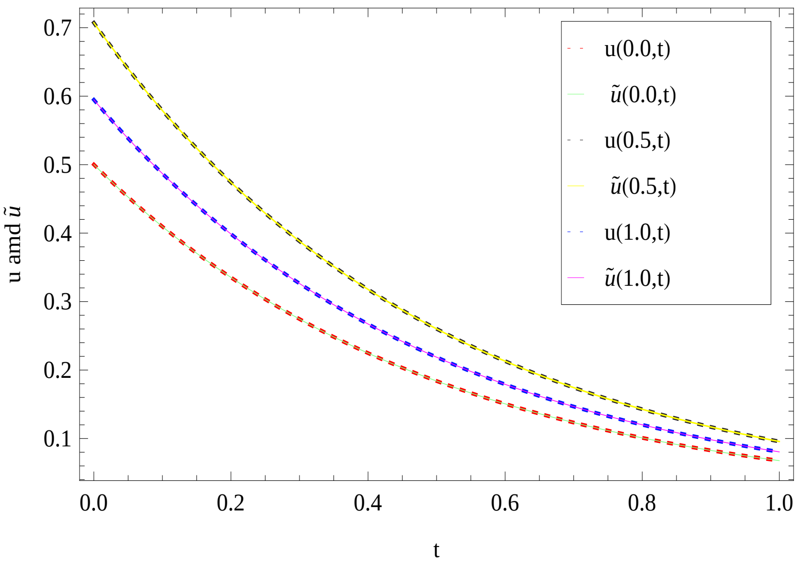

Fig. 2 – Temporal directional curves of exact and approximate solutions of problem (26) where $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$, and $N = M = 20$.

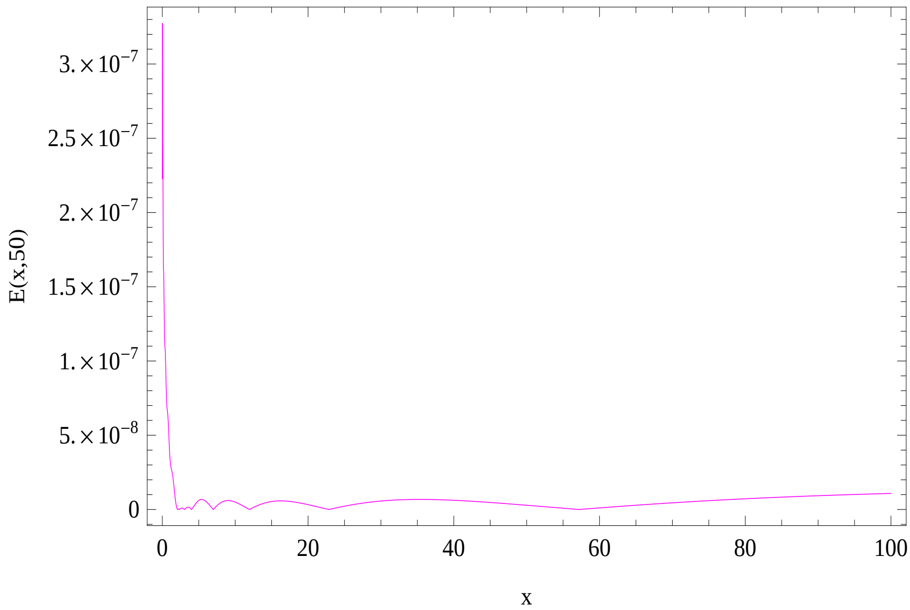

Fig. 3 – The absolute error of problem (26), using JR-GR-C method with $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$ at $N = M = 20$.

with the following conditions

$$u(0,t) = \text{sech}(2t), \quad u(x,0) = \text{sech}(x),$$
$$\lim_{x\to\infty} \partial_x u(x,t) = \lim_{t\to\infty} \partial_x u(x,t) = 0. \tag{30}$$

The solition solution of Eq. (29) is given by

$$u(x,t) = \text{sech}(2t + x), \quad (x,t) \in [0,\infty) \times [0,\infty). \tag{31}$$

Maximum absolute errors of problem (29) subject to (30) are presented in Table 3 using JR-GR-C method for different values of α_1, β_1, α_2, β_2, M and N. The special case $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$ is more accurate than the other two cases.

We see the matching of exact and approximate solutions curves in Fig. 4, with values of parameters listed in its caption. Meanwhile, we plot the absolute error curve in t direction in Fig. 5 at $x = 100$ using JR-GR-C method with $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = -\frac{1}{2}$, and $N = M = 20$.

Table 3.

Maximum absolute errors using JR-GR-C method for equation (29)

$\alpha_1=\beta_1$	$\alpha_2=\beta_2$	4	8	12	16	20
0	0	9.26×10^{-2}	4.85×10^{-3}	5.16×10^{-4}	8.32×10^{-5}	6.75×10^{-6}
$\frac{1}{2}$	0	9.75×10^{-2}	3.95×10^{-3}	4.95×10^{-4}	1.97×10^{-4}	1.28×10^{-5}
$-\frac{1}{2}$	$-\frac{1}{2}$	5.75×10^{-1}	3.15×10^{-2}	6.75×10^{-4}	2.70×10^{-4}	6.23×10^{-5}

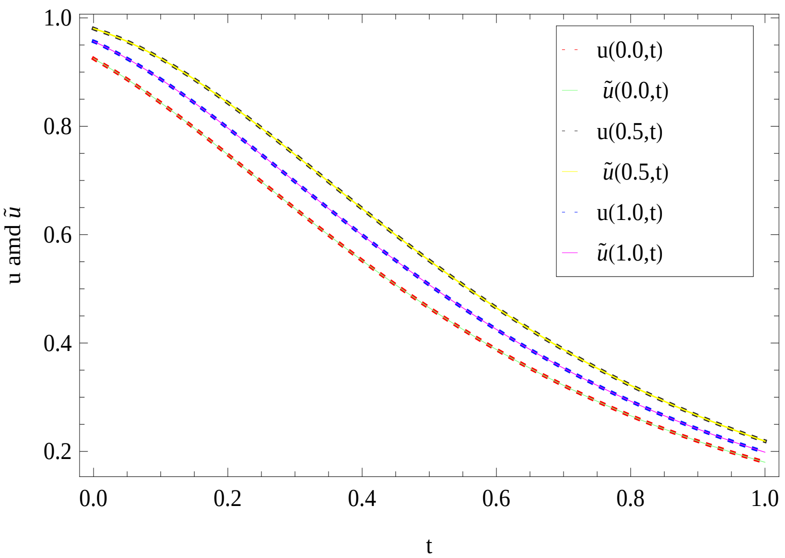

Fig. 4 – Temporal directional curves of exact and approximate solutions of problem (29) at x =0, 0.5 and 1, where $\alpha_1=\beta_1=\alpha_2=\beta_2=-\frac{1}{2}$, and $N=M=20$.

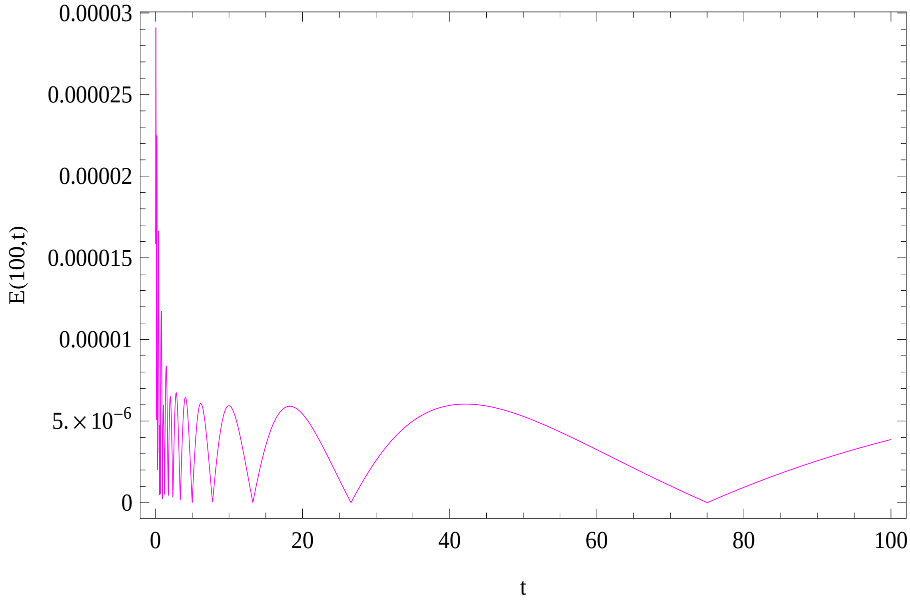

Fig. 5 – The absolute error of problem (29), using JR-GR-C method with $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = -\frac{1}{2}$, and $N = M = 20$.

5. Conclusion

We have proposed a new space-time collocation approach to spectrally solve the hyperbolic PDEs of second order in a semi-infinite domain. In this approach, the numerical solution was approximated by means of the Jacobi rational functions and the problem with its boundary conditions are collocated at Jacobi rational Gauss-Radau quadrature nodes. The mentioned problem was reduced into a system of algebraic equations in the expansion coefficients of the spectral solution.

The numerical results given in this paper demonstrated the good accuracy of the proposed method. During two numerical applications, we explained that the proposed method is simple and accurate. Indeed, while a limited number of Jacobi rational collocation nodes are adopted, very accurate numerical results are obtained. Finally, we can conclude that, the algorithm presented in this paper can be well suited for handling general linear and nonlinear PDEs in semi-infinite domains.

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